Introduction

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Tentative outline of these lectures:

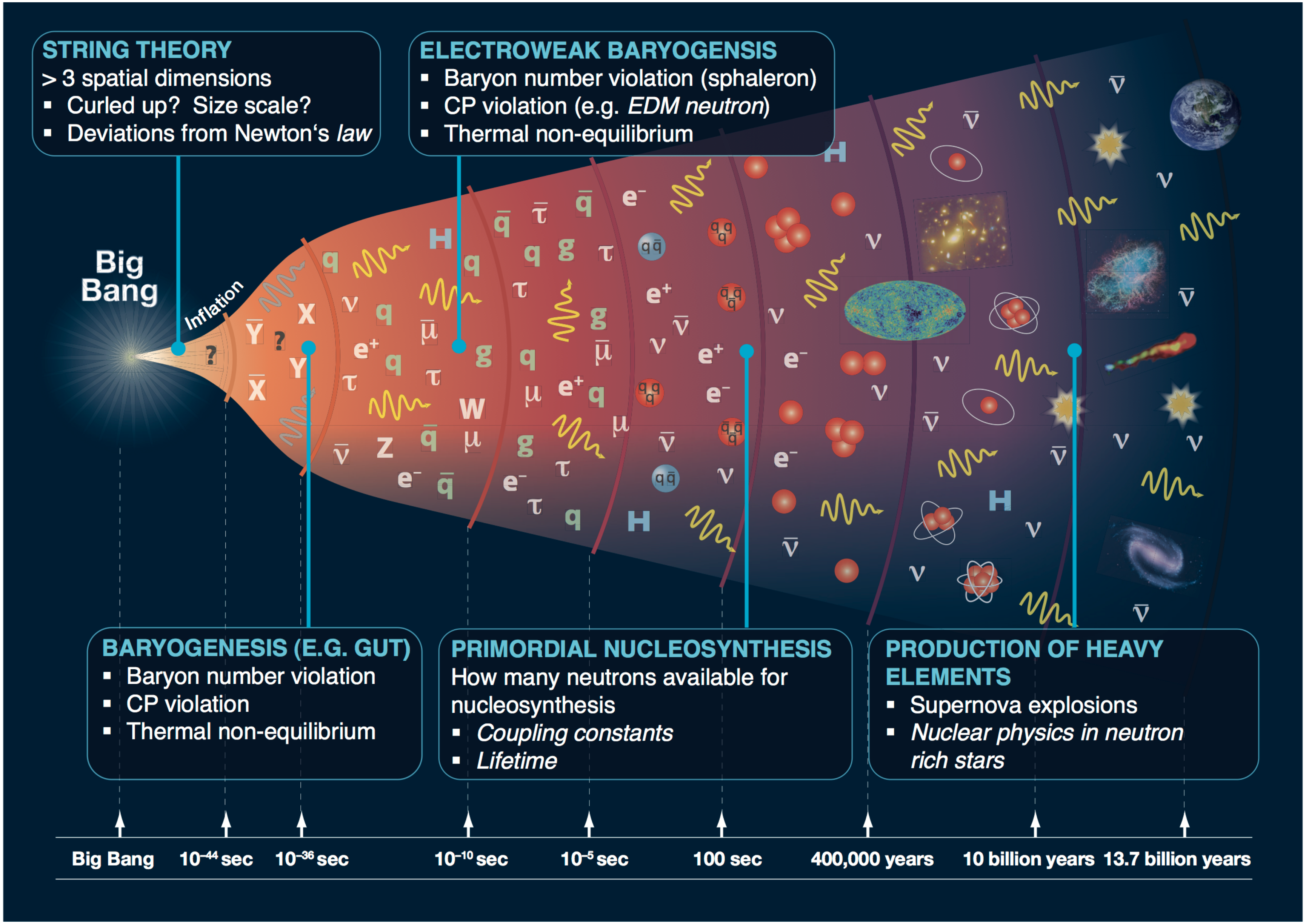
1. Today: introduction, key equations and techniques
2. Tomorrow and Sunday: mostly first order phase transitions

About me

- David Weir, email david.weir@helsinki.fi
- Postdoc working on numerical simulations of phase transitions in the early universe
- Previously:
  - 2014-16 Norway (Stavanger)
  - 2011-14 Helsinki (Finland)
  - 2007-11 Imperial College London (UK) [PhD]

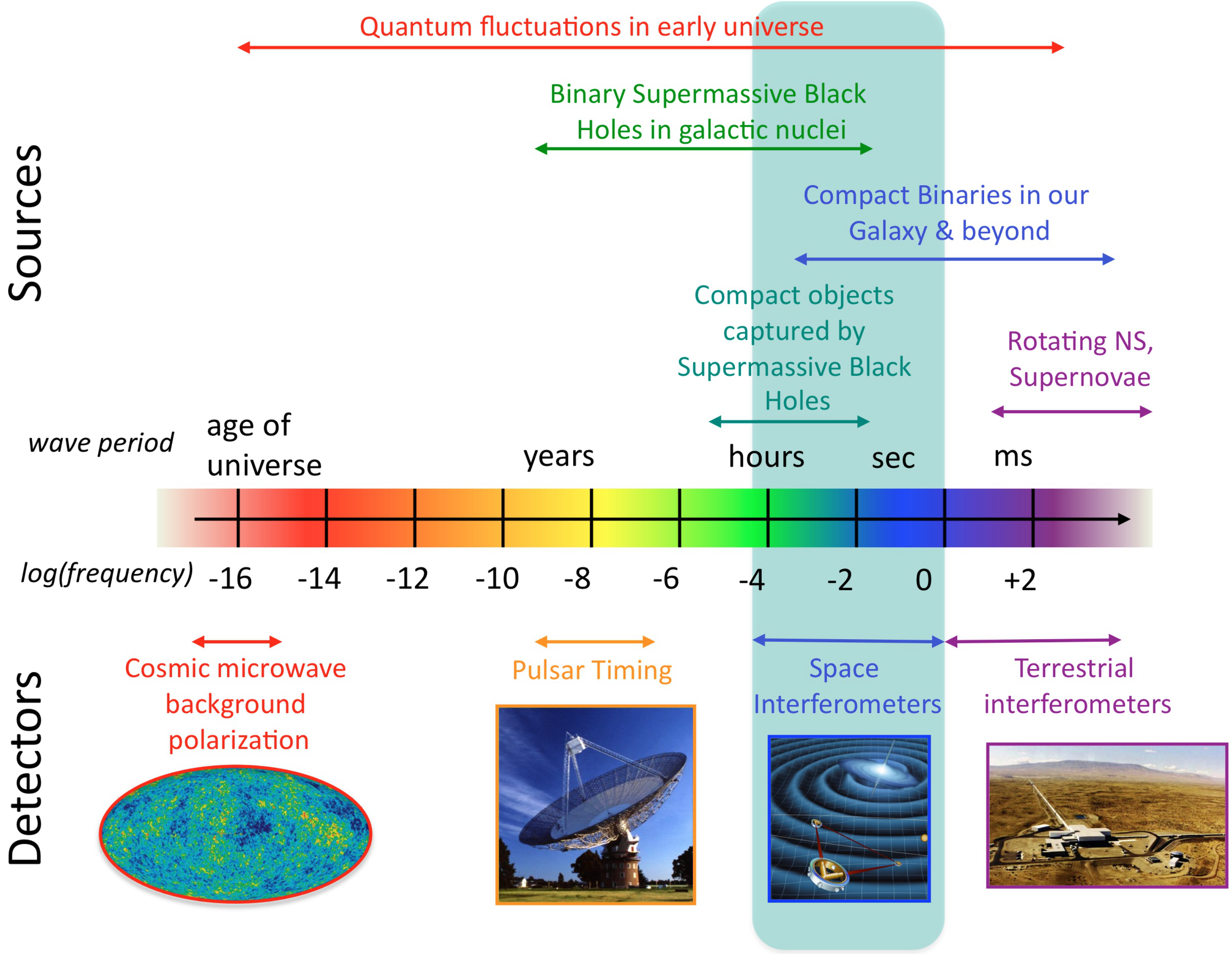
About you ...?

- Are you PhD students, postdocs, staff?
- Have you studied GR, gravitational waves, cosmology, particle physics?
- Are you working in this field?
- Please feel free to ask questions!
- And if you have requests, let me know.

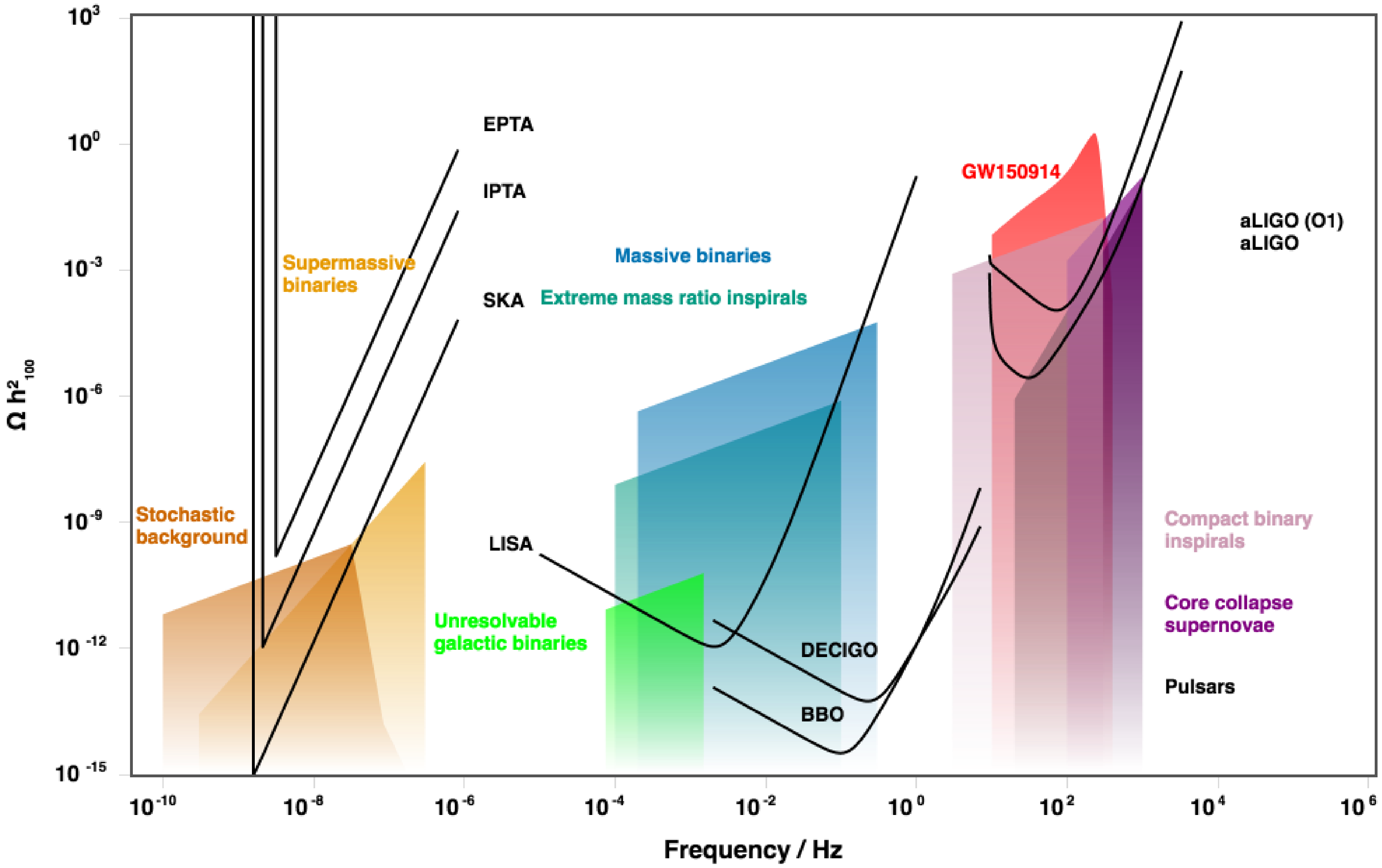


Source: arXiv:1205.2451

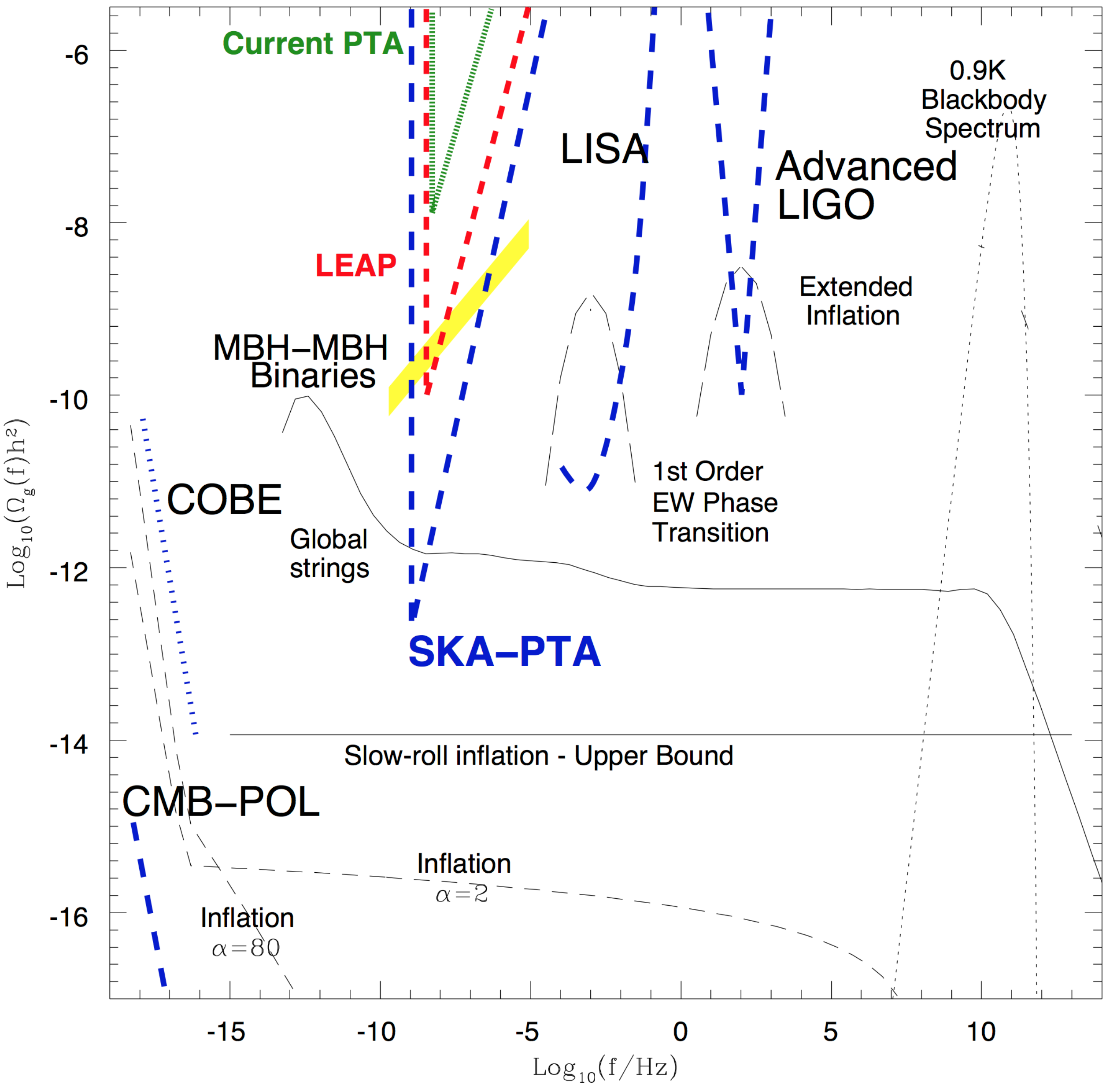
The Gravitational Wave Spectrum



Source: NASA



Source: GWplotter

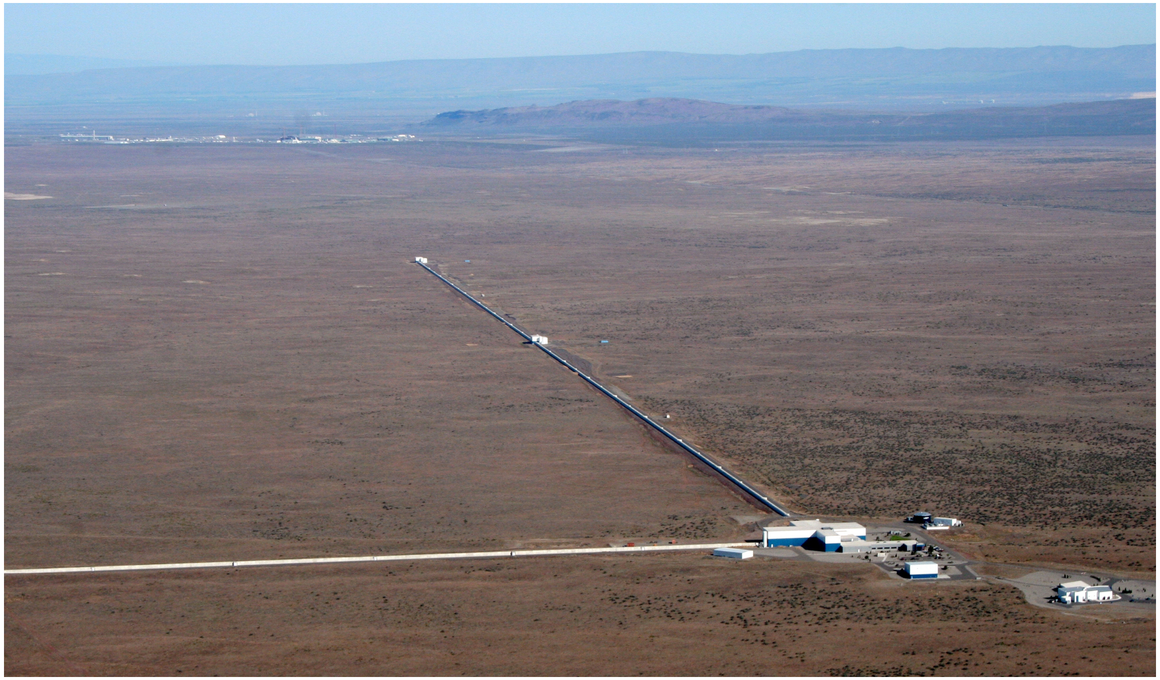


Source: Kramer and Stappers

LIGO


Ligo
BRAND
IN TOMATO SAUCE CHILI ADDED
IN TOMATO SAUCE CHILI ADDED
SARDINES
PRODUCT OF THE PHILIPPINES
NET WT 5.5 OZ. (155g)

LIGO at the Hanford Site

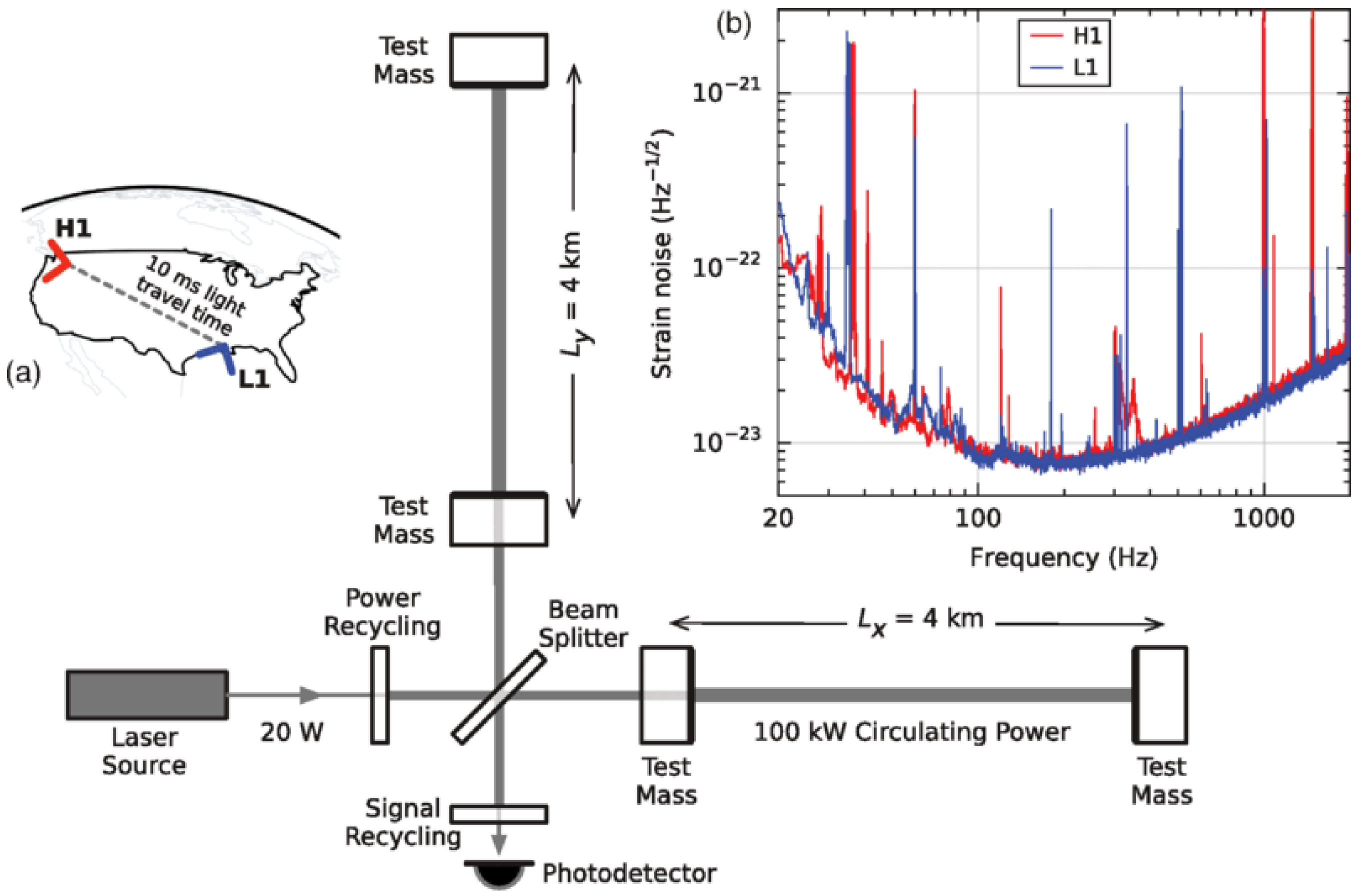


Source: (CC-BY-NC-ND) Prachatai

About LIGO

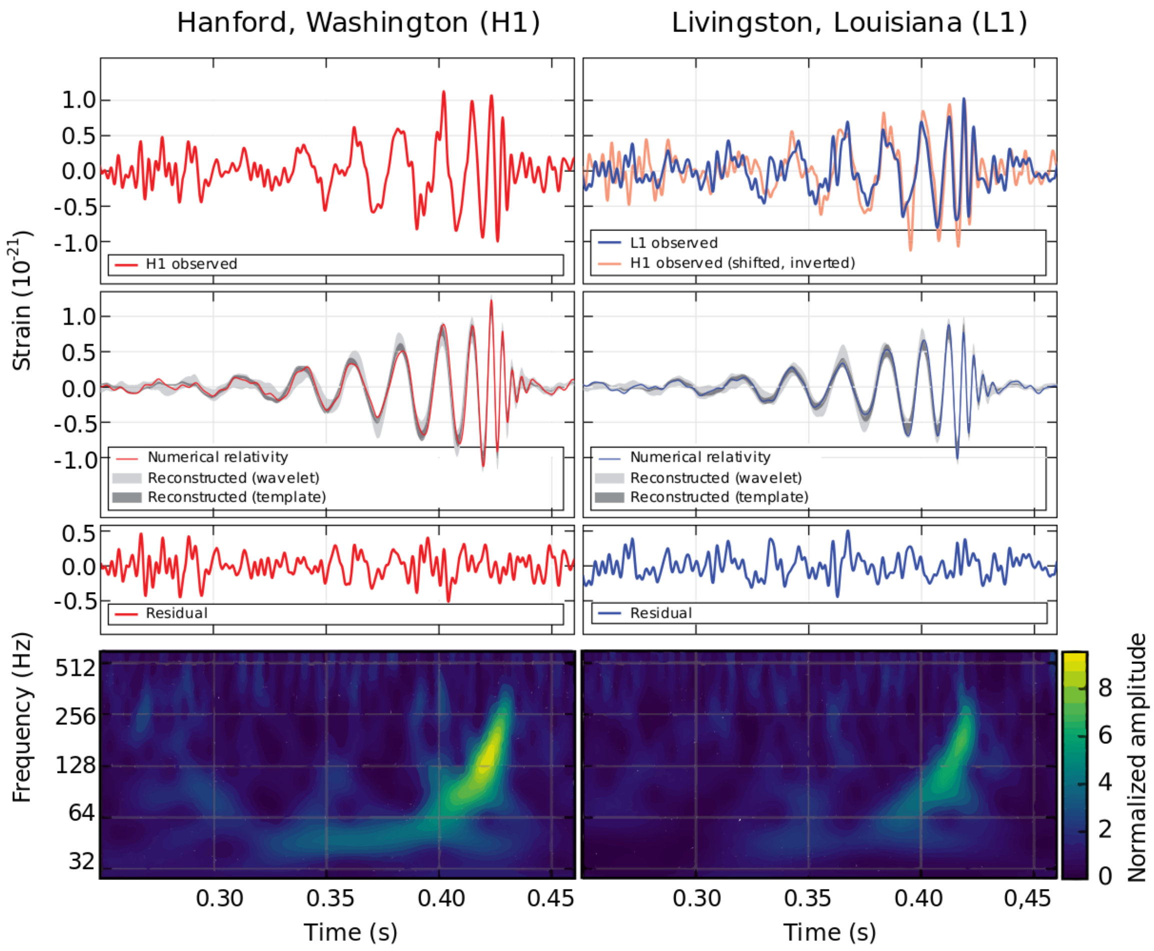
- 2 sites: Livingston, LA; Hanford Site, WA
- Cost: about a billion USD (most expensive/ambitious project ever funded by NSF)
- Each site: Michelson interferometers with 4km arms, 1064 nm Nd:YAG laser
- Each arm: Fabry-Pérot cavity (increases path length to equivalent of 280 trips)
- When a GW passes through: arms detune, photons emitted - signal
- Sister project in Europe: VIRGO

LIGO design

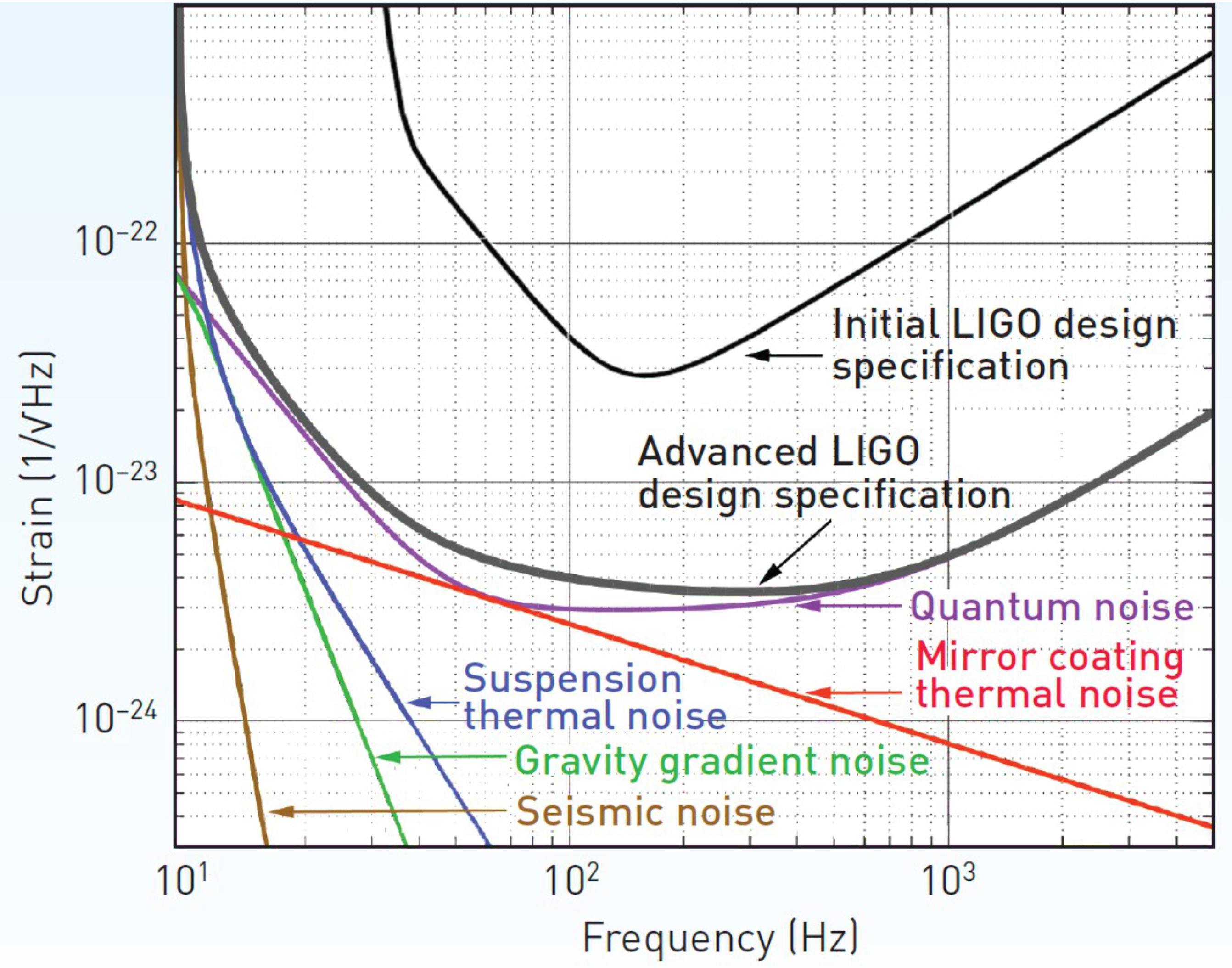


Source: (CC-BY) Phys. Rev. Lett. 116, 061102

First direct detection: GW150914



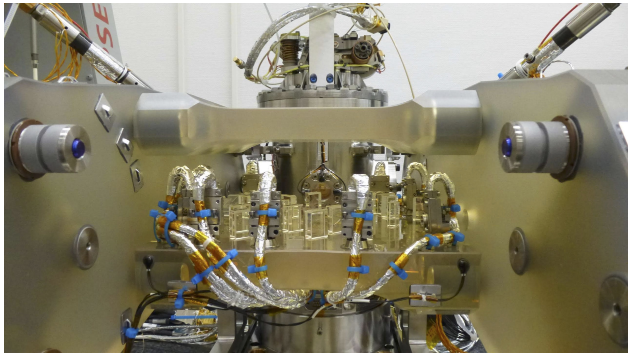
LIGO noise sources

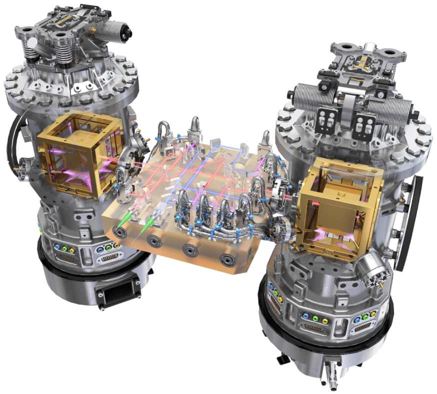


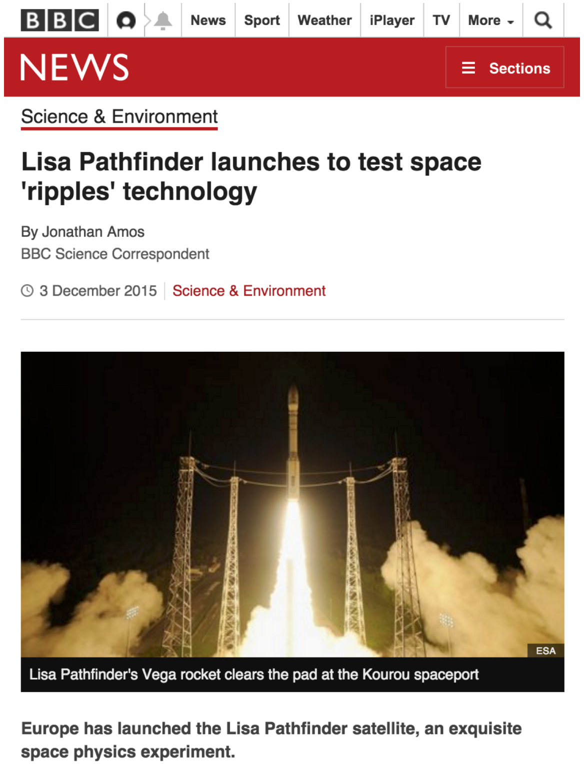
Source: LIGO via Optics and Photonics news

LISA (and LISA pathfinder)

To look at longer wavelengths, need to go into space!



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NEWS Sections

Science & Environment

Lisa Pathfinder launches to test space 'ripples' technology

By Jonathan Amos
BBC Science Correspondent

3 December 2015 | Science & Environment

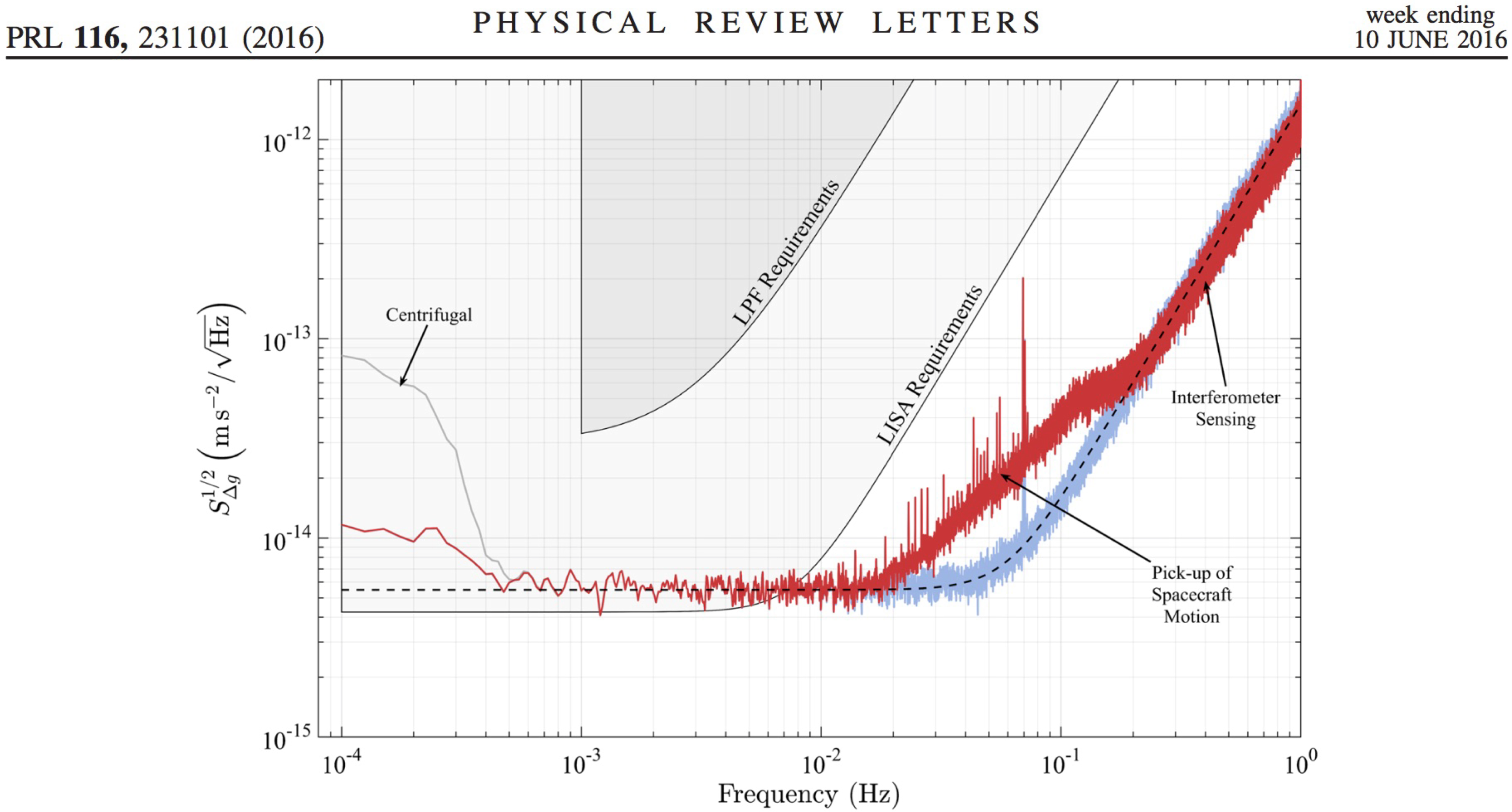
ESA

Lisa Pathfinder's Vega rocket clears the pad at the Kourou spaceport

Europe has launched the Lisa Pathfinder satellite, an exquisite space physics experiment.

Sources: LISA; BBC

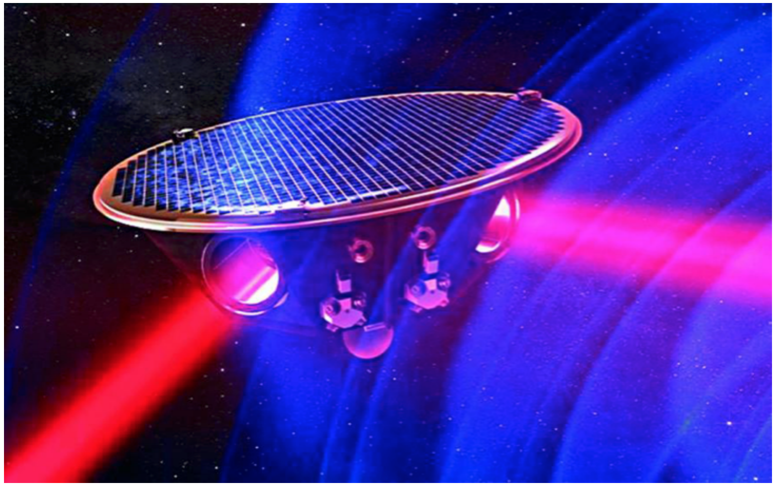
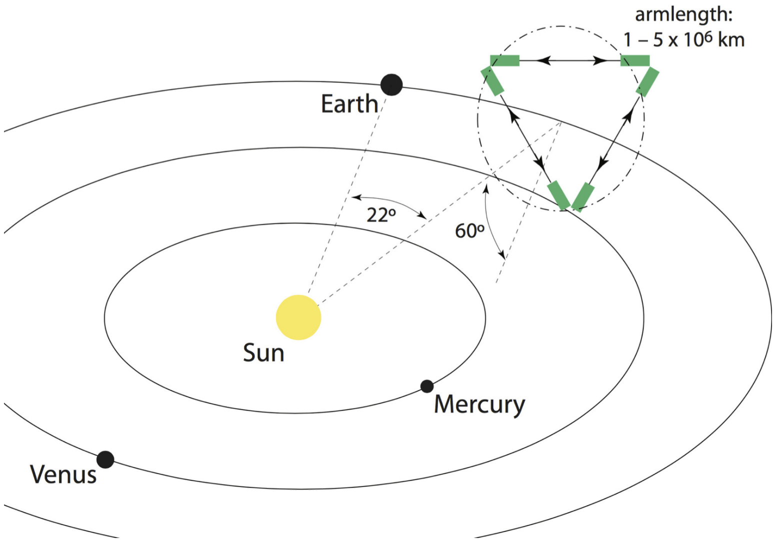
LISA Pathfinder



Exceeded design expectations by factor of five!

Source: (CC-BY) Phys. Rev. Lett. 116, 231101

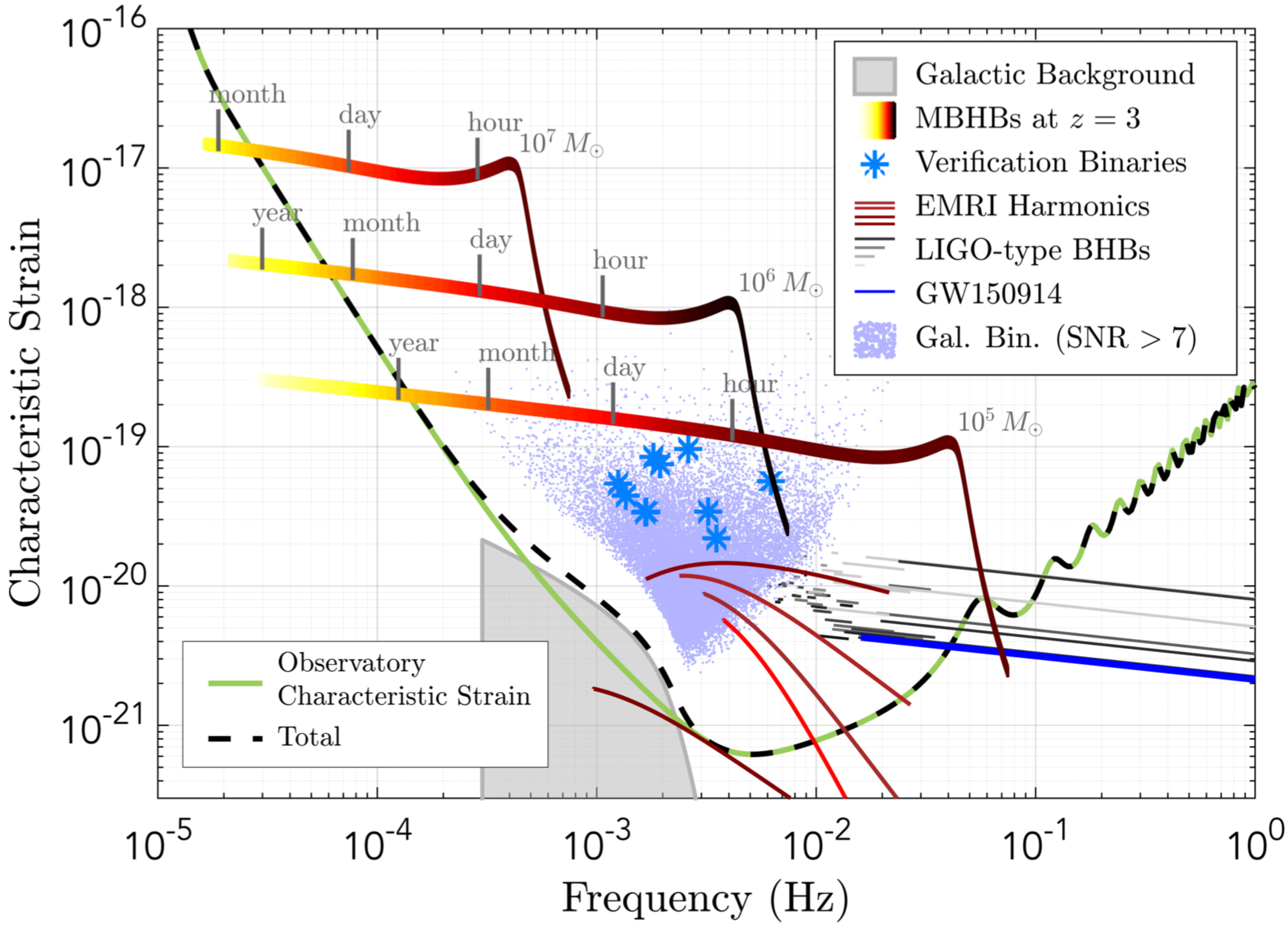
LISA mission profile

- LISA: three arms (six laser links), 2.5 M km separation
- Launch as ESA's third large-scale mission (L3) in (or before) 2034
- Proposal officially submitted earlier this year 1702.00786

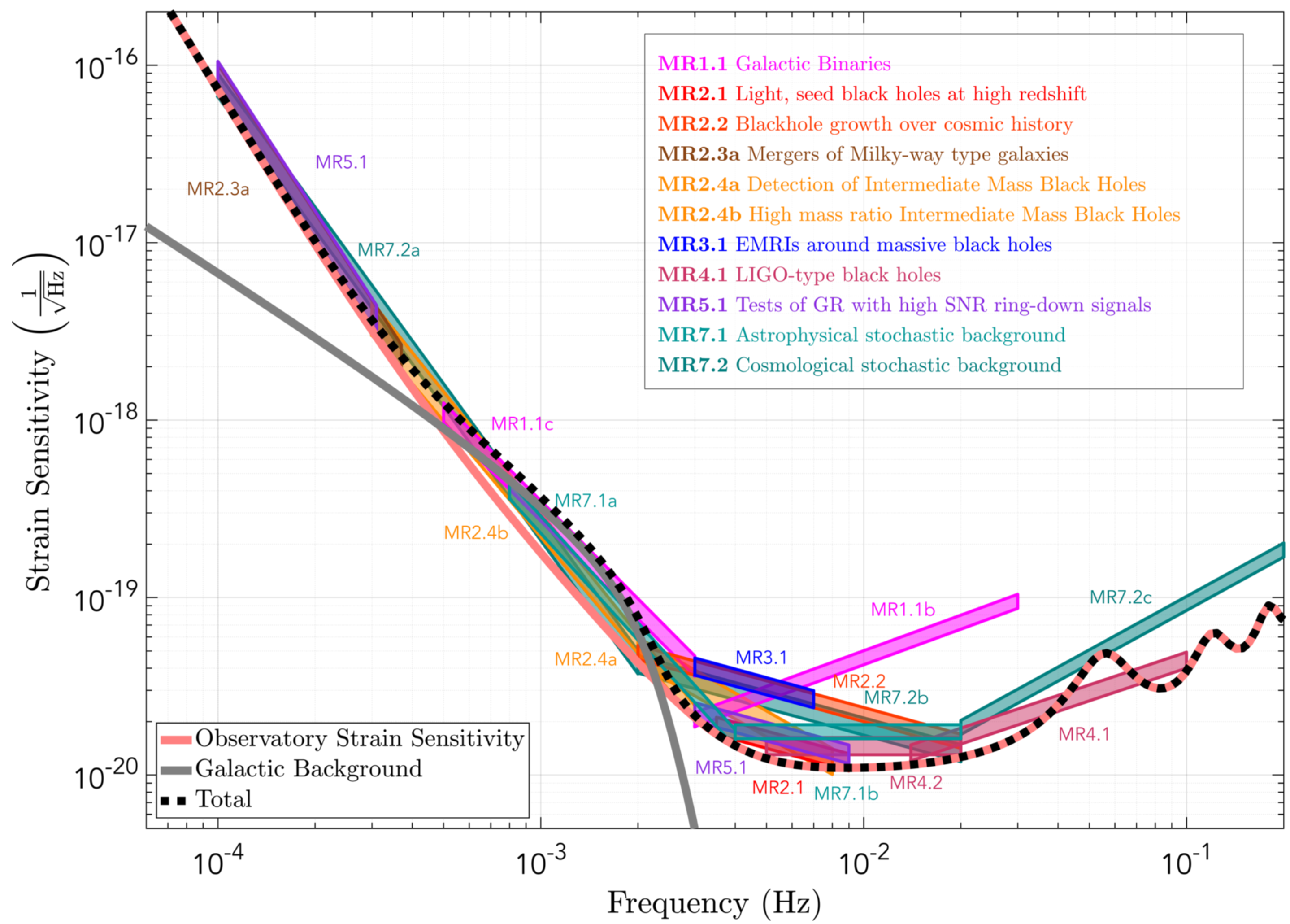
Source: LISA.

Possible signals



Source: LISA proposal.

Mission requirements



Source: LISA proposal.

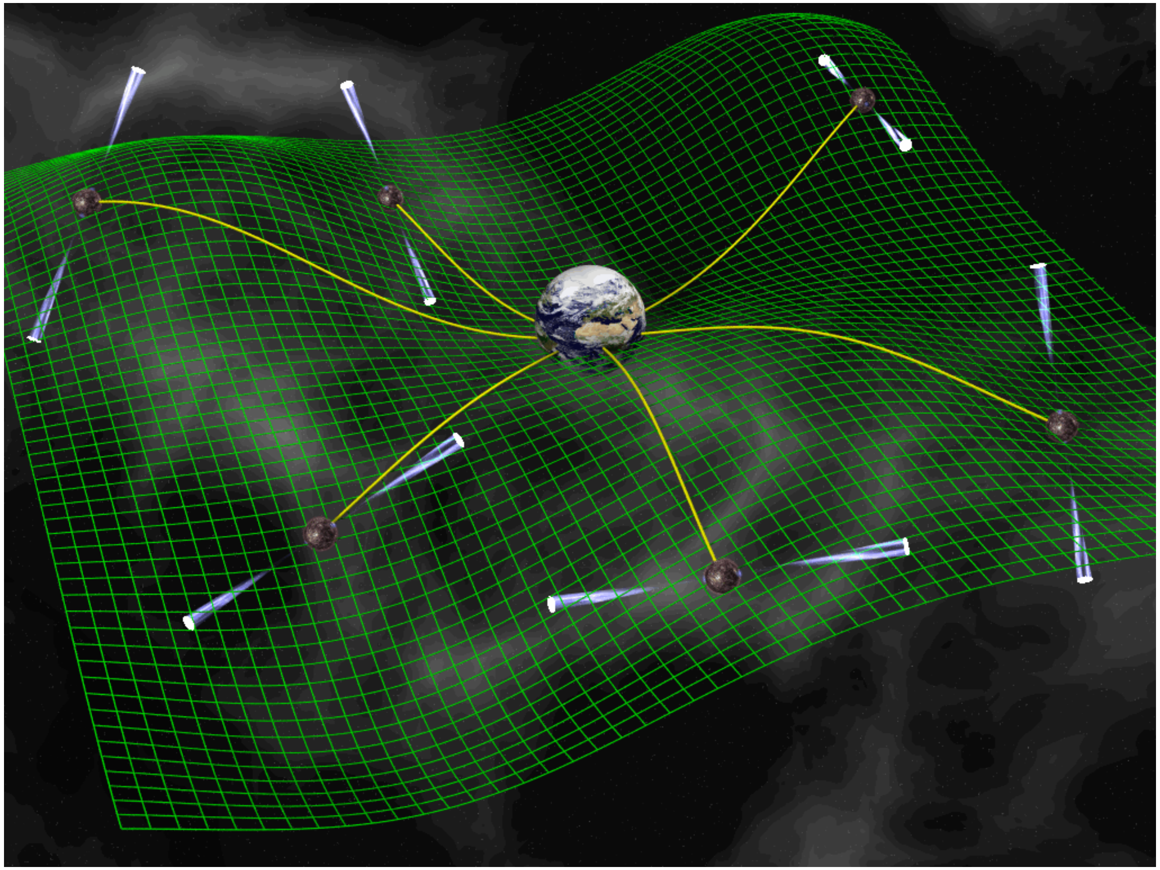
From the proposal:

SI7.2 : Measure, or set upper limits on, the spectral shape of the cosmological stochastic GW background

OR7.2: Probe a broken power-law stochastic background from the early Universe as predicted, for example, by first order phase transitions [21] (other spectral shapes are expected, for example, for cosmic strings [22] and inflation [23]). Therefore, we need the ability to measure $\Omega = 1.3 \times 10^{-11} \left(f/10^{-4}\,\mathrm{Hz}\right)^{-1}$ in the frequency ranges $0.1\,\mathrm{mHz} < f < 2\,\mathrm{mHz}$ and $2\,\mathrm{mHz} < f < 20\,\mathrm{mHz}$, and $\Omega = 4.5 \times 10^{-12} \left(f/10^{-2}\,\mathrm{Hz}\right)^{3}$ in the frequency ranges $2\,\mathrm{mHz} < f < 20\,\mathrm{mHz}$ and $0.02 < f < 0.2\,\mathrm{Hz}$.

Source: LISA proposal.

Pulsar timing arrays



Source: David Champion via MPIfR

About pulsar timing arrays

- Millisecond pulsars (MSPs) are very accurate clocks
- PTA projects time tens of MSPs every week or so
- Irregularities in the period of one pulsar might be due to a change in the proper distance to that pulsar because of a passing GW (or a similar effect at Earth).
- Measure correlations between multiple pulsars
  - Disentangle GWs at the pulsar end and at the Earth end
  - Remove effects of 'physics' in the neutron stars
- Study stochastic background at very low frequencies

More about pulsar timing arrays

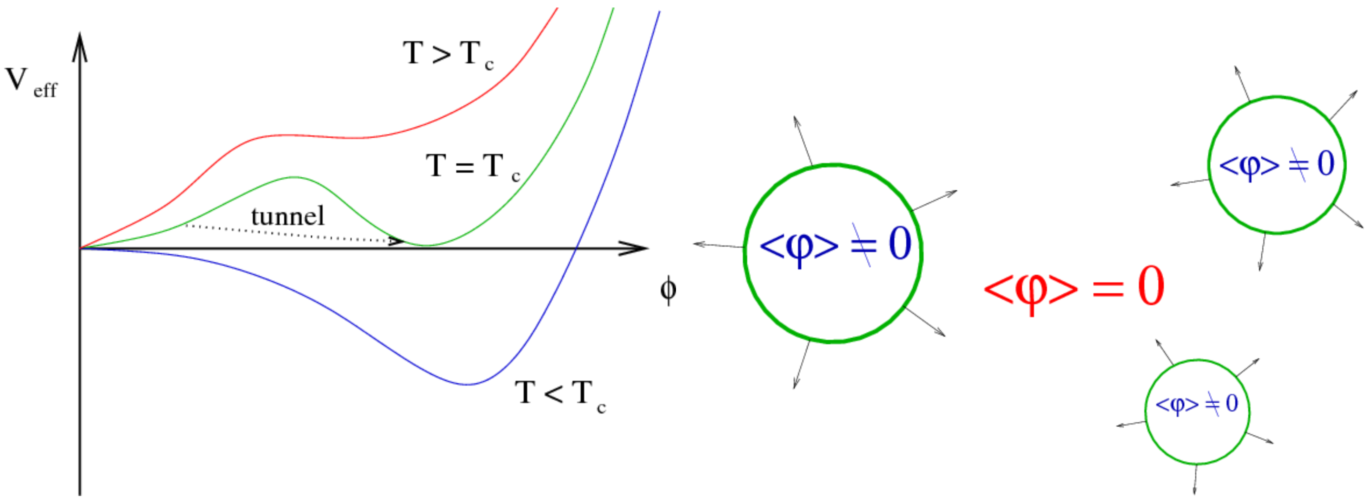
- They also set the some of *strongest* constraints on the equation of state of dense nuclear matter.
- Because MSPs are more accurate than atomic clocks, PTAs set the strongest constraints on solar system ephemerides
- Some of the biggest single users of radio telescope time

Source: NSF via Wikipedia

Cosmological sources of gravitational waves

Electroweak phase transition

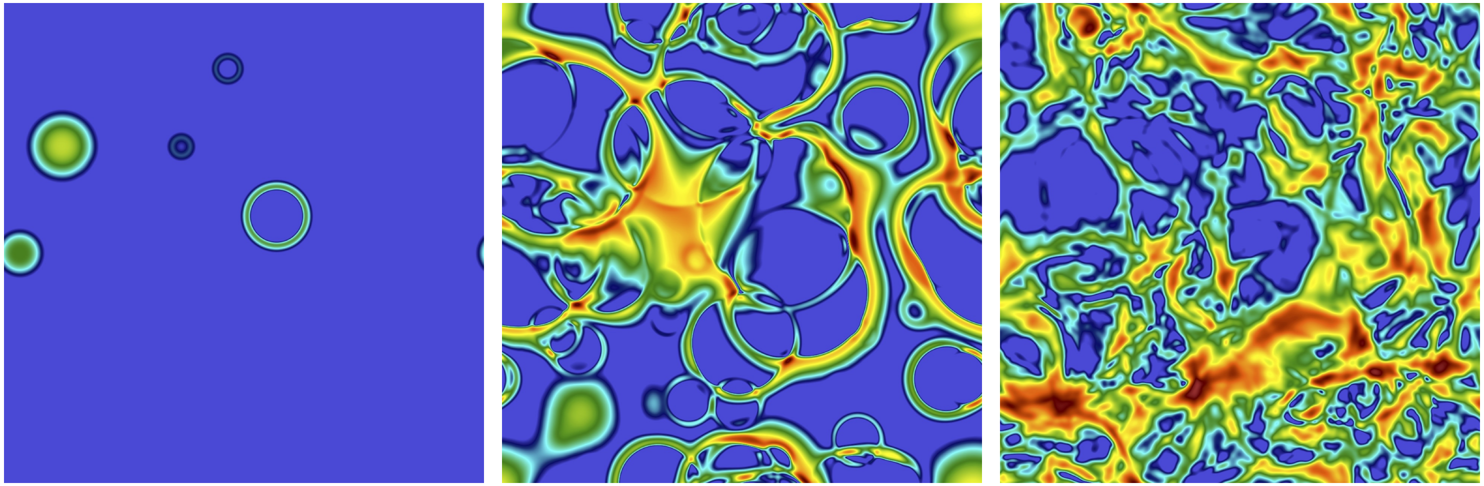
- This is the process by which the Higgs became massive
- In the minimal Standard Model it is gentle (crossover)
- It is possible (and theoretically attractive) in extensions that it would experience a first order phase transition



Source: Morrissey and Ramsey-Musolf

EWPT physics

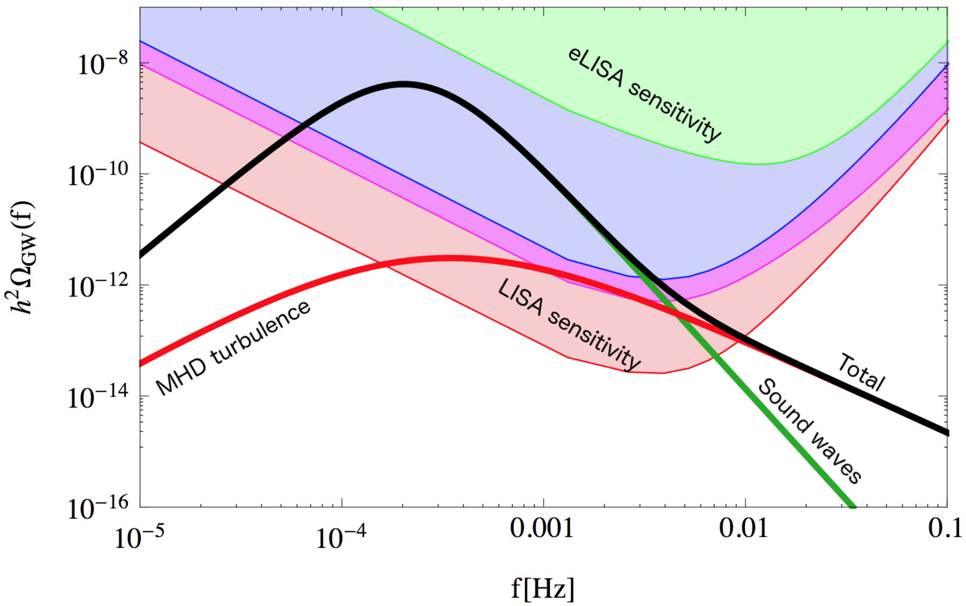
- Bubbles produced at a thermal phase transition will be the central focus of these lectures
- This would be a significant source of gravitational waves



Source: 1504.03291

Where to see it

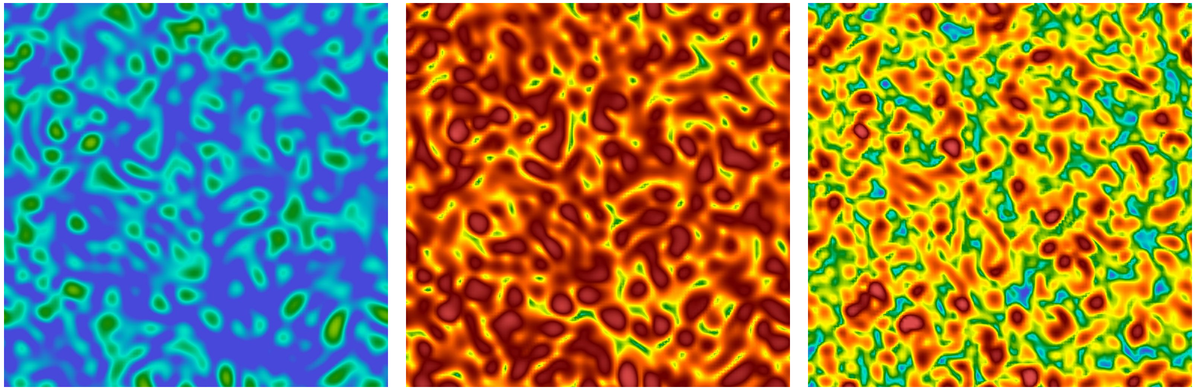
- The power spectrum is peaked at around the bubble radius, a fraction of the Hubble radius at the time of the transition.
- That corresponds to millihertz today, which means it is ideally placed for space-based detectors like LISA



Source: 1512.06239

The end of inflation

- As the inflaton reheated the universe, it created a lot of particles through violent processes.
- The inflation oscillating about the bottom of its potential would excite other particles to oscillate with characteristic frequencies given by their masses.
- These would be an efficient source of gravitational waves, but at high frequencies given by the mass of the field.



Source: 1506.06895.

Where to see it?

- Some scenarios are potentially observable at earth-based detectors such as advanced LIGO
- Otherwise, must wait for the more ambitious (post-LISA) space-based detectors (DECIGO, BBO, ...)

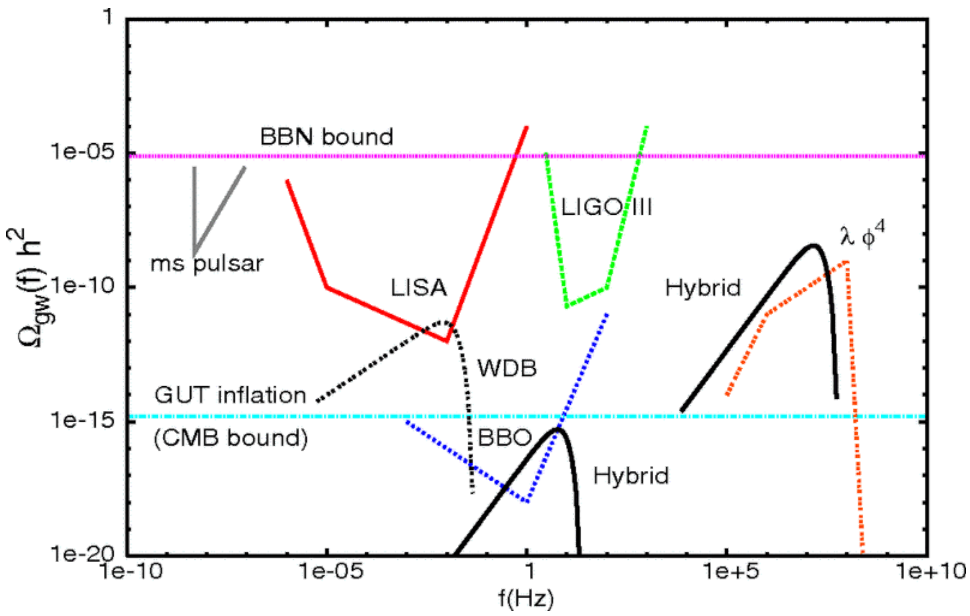


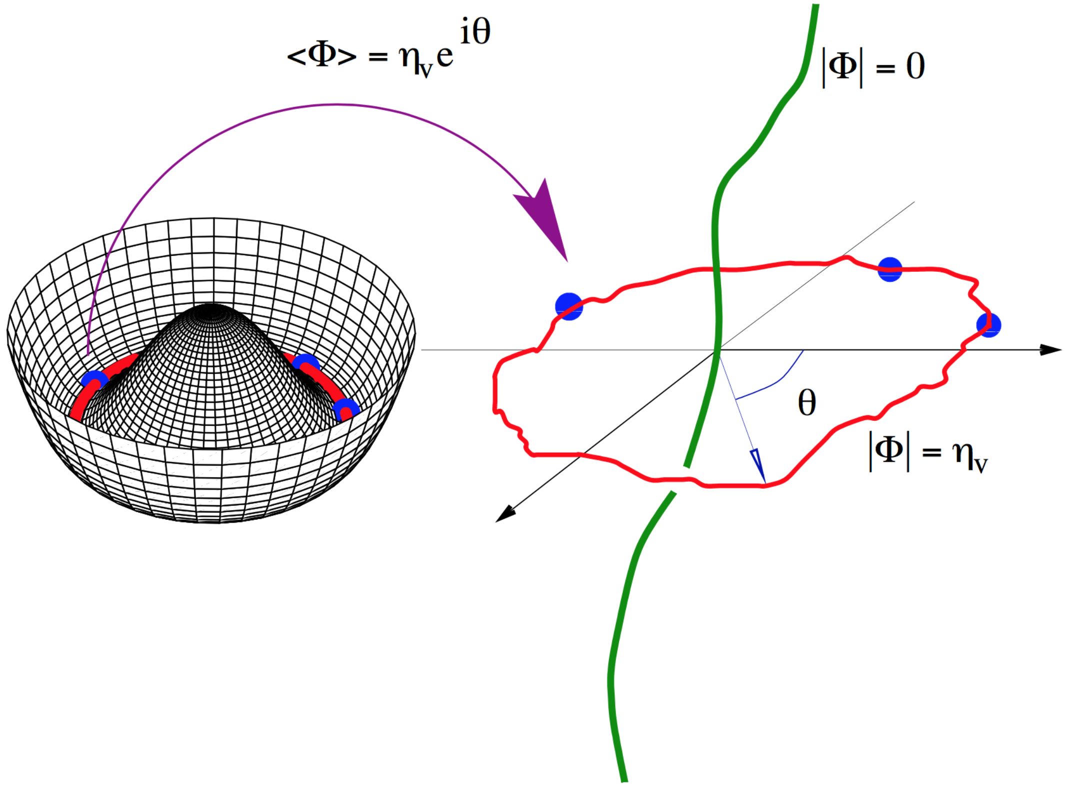
Figure source: Garcia-Bellido, Figueroa and Sastre

Cosmic strings

- Extended, long lived, very tense pieces of 'string' formed during symmetry breaking in the early universe
- They produce a scaling network, and lose energy principally by gravitational radiation
- They can also radiate particles, but exactly how much is an open research question!
- Loops of string are long-lived, emitting GWs over many wavelengths as they get smaller with time.

Topological stability

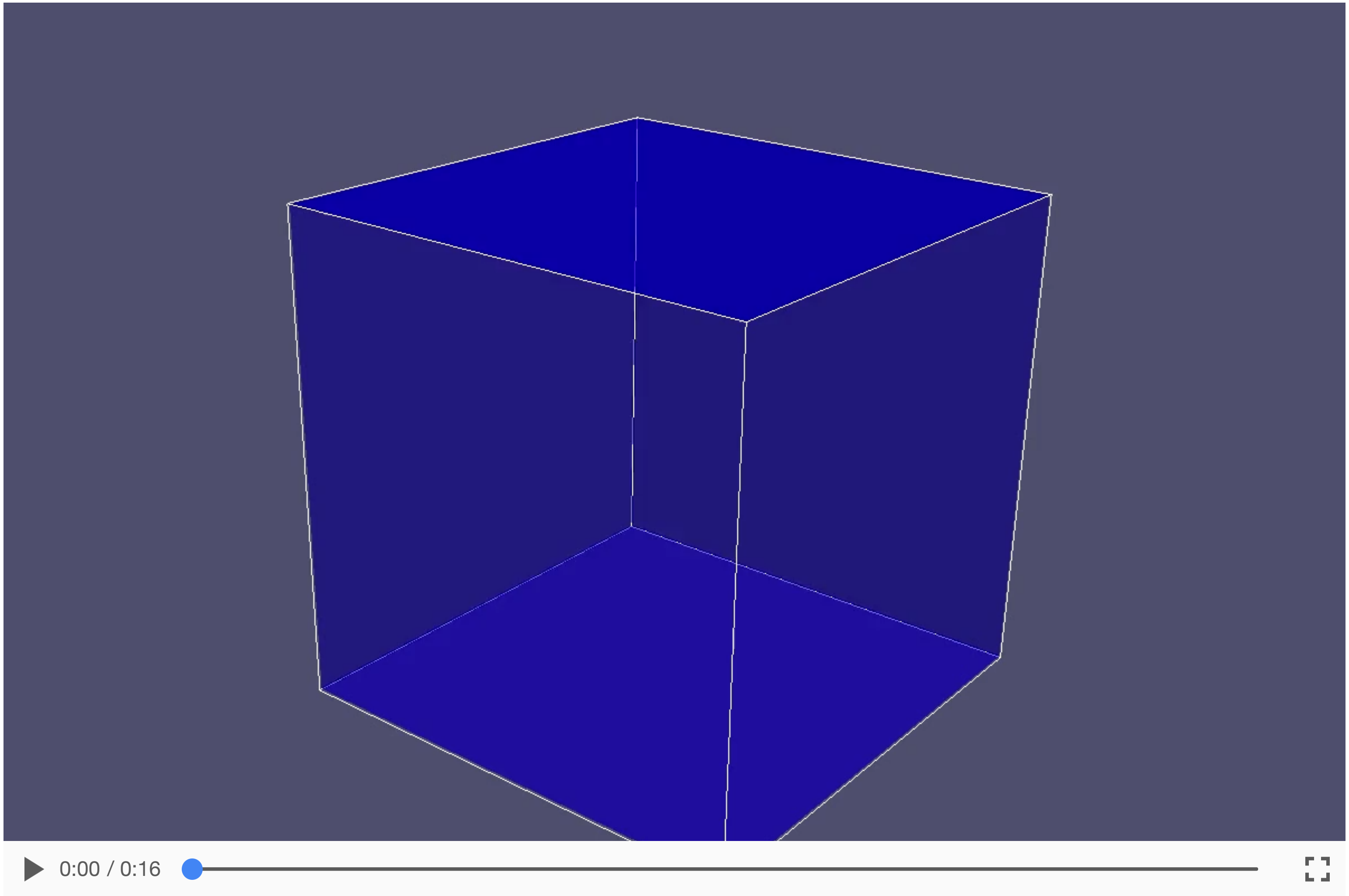
In field theory, cosmic strings are topologically stable:



Picture source: Ringeval

Can also be modelled by thin Nambu-Goto strings

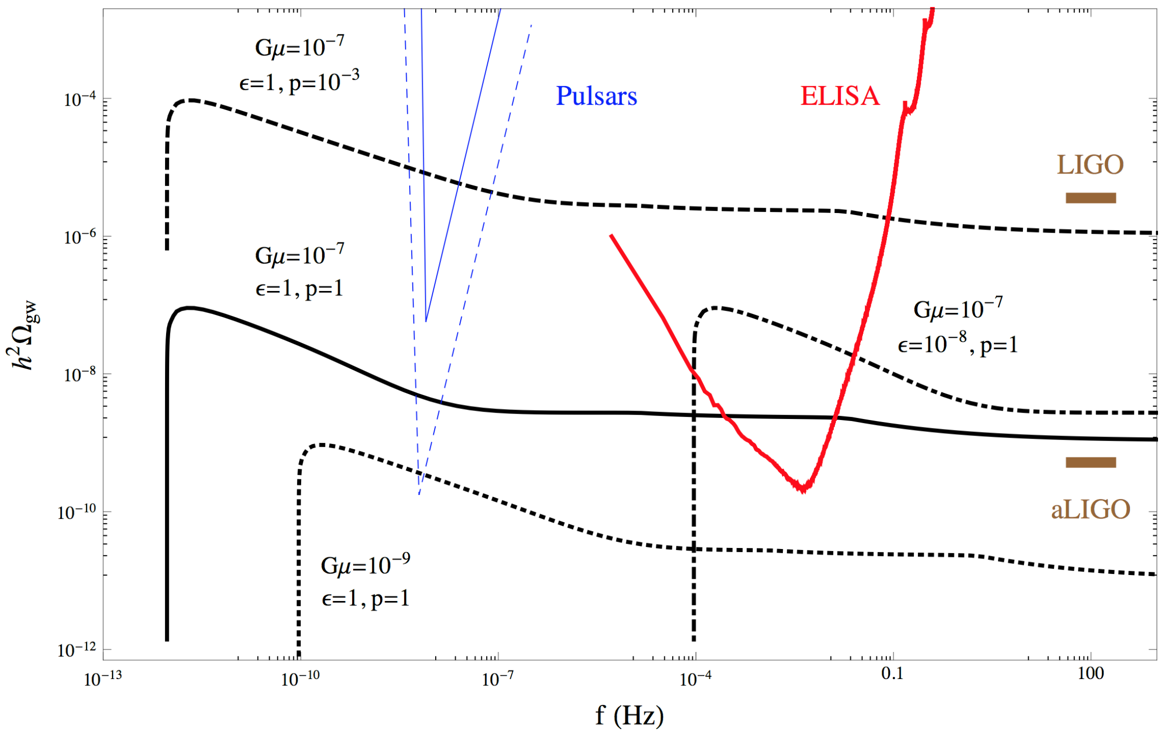
Cosmic strings movie 1



Credit: David Daverio

Where to see them?

- Pulsar timing arrays already place strong constraints on the energy scale on the energy of the strings.
- Improved PTA measurements will constrain $G\mu$ further, meaning the energy scale is lower



Picture source: Binétruy, Bohé, Caprini and Dufaux

Gravitational waves recap

Further reading:

- Review articles: Buonnano [follow this to some extent]
- Textbooks: Hartle; Schutz; Weinberg

Introduction

- The basic tools of general relativity are central ideas from differential geometry:
  - The metric, $g_{\mu\nu}$
  - The connections (also known as Christoffel symbols) $\Gamma^{\mu}_{\nu\rho}$
  - The Riemann tensor (or curvature tensor) $R_{\mu\nu\rho\sigma}$

The Einstein equations

- The Einstein equations are written
  $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$
  where
  - The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$
  - The Ricci tensor $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$
  - The stress-energy tensor is $T_{\mu\nu}$
- 16 equations as written, 6 independent equations
  - symmetry: 6 constraints
  - Bianchi identity: 4 constraints
- A gauge theory where the symmetries are *diffeomorphisms* (carefully constructed coordinate transformations)

Stress-energy tensor

Note that $T_{\mu\nu}$ looks schematically like

$$T_{\mu\nu} = \left(\begin{array}{c|c} \text{[energy density]} & \text{[energy flux]} \\ \hline & \\ \text{[momentum density]} & \text{[pressure and stress]} \\ & \end{array} \right)$$

Linearised gravity

- As we said, gravity is a gauge theory where the symmetry is diffeomorphism invariance.
- This means it is invariant under coordinate transformations
$$x^\mu \to x'^\mu(x)$$
which are invertible, and differentiable.
- Under a coordinate transformation the metric transforms as
$$g_{\mu\nu}(x) = g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}.$$

Specialising to linear perturbations

- We now assume that we can write the metric as
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$
where $\eta_{\mu\nu}$ is the Minkowski metric
$$\mathrm{diag}(-1, +1, +1, +1)$$
- Even with this assumption, there is an invariance under transformations of the form
$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$
provided that $|\partial_\mu \xi_\nu| \leq |h_{\mu\nu}|$

Wave equation

- By substituting
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
in the Einstein equations
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$
and retaining only terms at linear order we will obtain a wave-like equation for $h_{\mu\nu}$.

The wave equation, simplified

- We can write it in a fairly clear manner by adopting the *trace-reversed metric perturbation*

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$$

- It then takes the form

$$\Box \bar{h}_{\nu\sigma} + \eta_{\nu\sigma}\partial^{\rho}\partial^{\lambda}\bar{h}_{\rho\lambda} - \partial^{\rho}\partial_{\nu}\bar{h}_{\rho\sigma} - \partial^{\rho}\partial_{\sigma}\bar{h}_{\rho\nu} + \mathcal{O}(h^2) = -\frac{16\pi G}{c^4} T_{\nu\sigma}$$

Lorenz gauge

- Now we impose Lorenz gauge
$$\partial_\nu \bar{h}^{\mu\nu} = 0$$
- This is equivalent to Lorenz gauge in electromagnetism:
$$\partial_\mu A^\mu = 0$$
- The effect is to cancel every term except
$$\Box \bar{h}_{\nu\sigma} = -16\pi G T_{\nu\sigma}$$
which indeed looks like a wave equation!

Summary (comparison to electromagnetism)

Source: le Tiec and Novak

Table 2. The gauge freedom of linearized gravitation is analogous to that of ordinary electromagnetism in flat spacetime.

	Electromagnetism	Linearized gravity
Generator	χ	ξ_a
Potential	A_a	h_{ab}
Gauge transfo.	$A_a \to A_a + \partial_a \chi$	$h_{ab} \to h_{ab} + 2\partial_{(a}\xi_{b)}$
Gauge invariant	$F_{ab} = \partial_{[a}A_{b]}$	$R_{abcd} = -\partial_c\partial_{[a}h_{b]d} + \partial_d\partial_{[a}h_{b]c}$
Lorenz gauge cond.	$\partial^a A_a = 0$	$\partial^a \bar{h}_{ab} = 0$
Conservation law	$\partial^a j_a = 0$	$\partial^a T_{ab} = 0$
Wave equation	$\Box A_a = -\mu_0 \, j_a$	$\Box \bar{h}_{ab} = -16\pi G \, T_{ab}$

Transverse traceless gauge

- There are still 6 components in $\bar{h}_{\mu\nu}$, only 2 of which are physical - the two polarisations
- To find these, we consider coordinate transformations that *also* satisfy Lorenz gauge, meaning
$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x); \qquad |\partial_\mu \xi_\nu| \leq |h_{\mu\nu}|$$
and $\Box \xi^\mu = 0$.
- $\Box \xi^\mu = 0$ gives 4 new constraints. First we choose:
$$\bar{h} = 0$$
i.e. make $\bar{h}^{\mu\nu}$ traceless [one constraint]
- (Note that with this constraint, the trace of $h^{\mu\nu}$ also disappears, so we no longer need the 'trace reversed' tensor!)

More constraints

- We choose our other 3 constraints to be:

$$h^{i0} = 0$$

meaning $\partial_0 h^{00} = 0$ too, and we can choose $h^{00} = 0$.

- Put another way, our 4 constraints are:

$$h^{00} = h^{0i} = 0$$

- The remaining spatial entries of h_{ij} must be **transverse**

$$\partial_i h^{ij} = 0$$

and **traceless**

$$h^{ii} = 0.$$

Travelling waves

- A plane wave in the z-direction looks like
$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\,\omega\left(t - \frac{z}{c}\right)$$
Note that the polarisation components are perpendicular to the direction of travel.
- For a more general travelling wave with wave vector **k**, transverse traceless simplifies to
$$\hat{\mathbf{k}}^i h_{ij}^{\mathrm{TT}} = 0.$$

Projection

- If we have a general metric perturbation, we can use a tensor version of the usual 'transverse' projector:

$$\Lambda_{ij,lm} \equiv P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm}$$

- Here we have

$$P_{ij} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j$$

- And this satisfies

$$P_{ij} = P_{ji}, \quad \hat{\mathbf{k}}^i P_{ij} = 0, \quad P_{ij}P^{jk} = P_i^k, \quad P_{ii} = 2$$

- In other words, it projects out the rotational part of a vector field.

Projection of a general h_{ij}

- If we have a general h_{ij} which is in Lorenz gauge but does not otherwise satisfy the transverse traceless (TT) requirements, we can project out the TT parts:

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,lm} h^{lm}$$

- Why do we want to do this?
  - We may want to study the polarisation of the gravitational waves (but for stochastic, cosmological sources, these rarely matter)
  - More importantly, as h_{ij}^{TT} contains *only* the propagating degrees of freedom, measures of energy and power *must* be done with it.

Effective stress-energy tensor

- Step back and consider a more general background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

- What is the background and what is the perturbation?
  - $\bar{g}_{\mu\nu}$ has a scale L_B
  - $h_{\mu\nu}$ have a wavelength $\lambda \ll L_\mathrm{B}$.
- Or, in frequency:
  - $\bar{g}_{\mu\nu}$ only has frequencies up to f_B
  - $h_{\mu\nu}$ has frequencies $f \gg f_\mathrm{B}$.
- The overall effect is that, even if $\bar{g}_{\mu\nu}$ has come curvature, it looks slowly varying to the gravitational waves.

The Isaacson argument

- We can now split the Ricci tensor up as follows:
$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \ldots$$
  - $\bar{R}_{\mu\nu}$ is the background Ricci tensor
  - $R^{(1)}_{\mu\nu}$ are high frequency modes at linear order in h
  - $R^{(2)}_{\mu\nu}$ is everything at quadratic order in h
- If we average over a volume bigger than λ but smaller than L_{B} then we can use the Einstein equations to introduce $t_{\mu\nu} = -\frac{1}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle$
- And $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi G\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)$.

The Isaacson expression

- We end up with (in arbitrary gauge)

$$t_{\alpha\beta} = \frac{1}{32\pi G}\Big\langle \partial_\alpha \bar{h}_{\mu\nu}\partial_\beta \bar{h}^{\mu\nu} - \frac{1}{2}\partial_\alpha \bar{h}\partial_\beta \bar{h}$$

$$-\partial_\nu \bar{h}^{\mu\nu}\partial_\beta \bar{h}_{\mu\alpha} - \partial_\nu \bar{h}^{\mu\nu}\partial_\alpha \bar{h}_{\mu\beta}\Big\rangle$$

which, for TT, reduces to

$$t_{\alpha\beta} = \frac{1}{32\pi G}\left\langle \partial_\alpha h^{\mathrm{TT}}_{\mu\nu}\partial_\beta h^{\mu\nu}_{\mathrm{TT}}\right\rangle$$

which is known as the *Isaacson tensor*.

Energy density in gravitational waves

- Now that we have the effective stress-energy tensor, the energy density in gravitational waves is

$$\rho_{\mathrm{GW}} \equiv t_{00} = \frac{1}{32\pi G}\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \rangle$$

- By Fourier transforming this expression, define the GW power *per logarithmic frequency interval*

$$\frac{\mathrm{d}\,\rho_{\mathrm{GW}}(\mathbf{k})}{\mathrm{d}\,\log k} = \frac{1}{32\pi G}\frac{k^3}{2\pi^2}\left\langle \dot{h}_{ij}^{\mathrm{TT}}(\mathbf{k})\dot{h}_{ij}^{\mathrm{TT}}(-\mathbf{k})\right\rangle$$

- These two quantities are what cosmologists most widely quote in papers about gravitational waves, particularly the results of numerical simulations.

Typical assumptions in these lectures

- Minkowski or FRW spacetime: no, or isotropic expansion
  - Physics on timescales much shorter than expansion
  - All gravitational waves sourced by sub-Horizon physics
- Homogeneous, stochastic, isotropic source
  - True for most cosmological sources
  - May not be true for, e.g. cosmic string cusps

Results in context

How does this work in a cosmological simulation?

1. Evolve Lorenz-gauge wave equation in position space
$$\nabla^2 h_{ij}(\mathbf{x}, t) - \frac{\partial}{\partial t^2} h_{ij}(\mathbf{x}, t) = 8\pi G T_{ij}^{\text{source}}(\mathbf{x}, t)$$
during simulation, using relevant T_{ij}^{source} of 'source system'.
2. Projection to TT-gauge requires expensive Fourier transform, so only project when measurement desired:
$$h_{ij}^{\text{TT}}(\mathbf{k}, t_{\text{meas}}) = \Lambda_{ij,lm}(\hat{\mathbf{k}}) h^{lm}(\mathbf{k}, t)$$
3. Measure energy density (or power) in gravitational waves
$$\rho_{\text{GW}}(t_{\text{meas}}) = \frac{1}{32\pi G} \left\langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right\rangle$$
4. Redshift to present day.

Other techniques

- Two other approaches are sometimes seen in numerical cosmology:
  - **Quadrupole approximation** - as we will see, this is a poor approximation for bubble collisions we will be studying, but it still provides insight
  - **"Weinberg formula"** - this gives a clean time-domain formula where the stress-energy tensor takes simple forms
- We will look at these, and the properties of general sources, next.

Compact, distant sources

- This is also relevant, e.g. for colliding pairs of bubbles.
- Start from the Lorentz-gauge wave equation

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

- Solve in position space with retarded Green's functions

$$\bar{h}_{\mu\nu}(x) = -16\pi G \int \mathrm{d}^4 x' \, G(x - x') T_{\mu\nu}(x').$$

- If we now specialise to TT gauge and write in terms of the *retarded time* $t - |\mathbf{x} - \mathbf{x}'|$,

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \Lambda_{ij,lm}(\hat{\mathbf{n}}) \, 4G \int d^3 x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} \times T_{lm}\left(t - |\mathbf{x} - \mathbf{x}|; \mathbf{x}'\right)$$

Far field approximation

- And if we are also far from the source (where $T_{lm}(t, \mathbf{x}) \neq 0$),
$$|\mathbf{x} - \mathbf{x}'| \approx r - \mathbf{x}' \cdot \hat{\mathbf{n}}$$
and
$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) \approx \frac{1}{r} 4G \Lambda_{ij,lm}(\hat{\mathbf{n}}) \int_{\text{source}} d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \times T_{lm}\left(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}; \mathbf{x}'\right)$$
- We will assume(!) that velocities inside the source are non-relativistic
- In other words gravitational waves have lower frequencies ω than the source diameter:
$$\omega \mathbf{x}' \cdot \hat{\mathbf{n}} \ll 1$$

Multipole expansion

- We can write the source using a Fourier transform

$$T_{lm}\left(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}; \mathbf{x}'\right) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} T_{lm}(\omega, \mathbf{k}) e^{-i\omega\left(t-r+\mathbf{x}'\cdot\hat{\mathbf{n}}\right)+i\mathbf{k}\cdot\mathbf{x}'}$$

  and the leading order term expanding in $\omega\mathbf{x}' \cdot \hat{\mathbf{n}}$ is

$$T_{lm}\left(t - r + \mathbf{x}' \cdot \hat{\mathbf{n}}; \mathbf{x}'\right) \approx \int \frac{\mathrm{d}^4 k}{(2\pi)^4} T_{lm}(\omega, \mathbf{k})$$

  This is the **quadrupole** term.

Quadrupole source

- To leading order, then, the metric perturbation is
$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) \approx \frac{1}{r}\, 4G\, \Lambda_{ij,lm}(\hat{\mathbf{n}}) \int d^3x\, T^{lm}\,(t - r, \mathbf{x}) + \ldots$$
- This is zero for a spherically symmetric source (or linear superposition of spherically symmetric sources)
  - Vacuum fluctuations at the end of inflation
  - Freshly nucleated bubbles in the early universe
  - Isolated massive objects
- Need some non-spherical dynamics:
  - Particle resonances
  - Bubbles colliding
  - Binary compact massive objects

Why we must go beyond the quadrupole approximation

- In the early universe, velocities within the source are not small, and the gravitational waves are typically the same scale as the bubbles.

Weinberg formula

- So far, we have seen two methods of computing h_{ij}
  - Numerically solving the equation of motion
    $$\nabla^2 h_{ij} - \frac{\partial}{\partial t^2} h_{ij} = 8\pi G T_{ij}$$
    e.g. during a simulation
  - Using the quadrupole approximation if the wavelength of the gravitational waves is long compared to the size of the source(s)
- However, sometimes can simplify the source so that it is simple in Fourier space, and we do not need to do the quadrupole approximation
- Then we can use the Weinberg formula

Using the Weinberg formula

- For radiation in a direction $\hat{\mathbf{k}}$ and frequency ω, the power spectrum per logarithmic frequency interval, per unit solid angle,

$$\frac{d\rho_{\mathrm{GW}}}{d\log\omega\, d\Omega} = 2G\omega^3 \Lambda_{ij,lm}(\hat{\mathbf{k}}) T^*_{ij}(\hat{\mathbf{k}}, \omega) T_{lm}(\hat{\mathbf{k}}, \omega)$$

- One application of this is the 'envelope approximation', which we shall revisit later.

Useful insight 1

Source: Dufaux, Felder, Kofman, Navros

- Begin with the momentum-space Green's function expression (assume source off at $t' < 0$)

$$h_{ij}^{\mathrm{TT}}(\mathbf{k}, t) = 16\pi G \, \Lambda_{ij,lm} \int_0^t \mathrm{d}t' \, \frac{\sin[k(t - t')]}{k} T_{lm}(\mathbf{k}, t')$$

- If the source is slowly varying in space at low $\mathbf{k}$:

$$T_{lm}(\mathbf{k}) \to \text{const.}; \qquad k \ll k_{\max}$$

(equivalent to the quadrupole approximation) we get

$$h_{ij}^{\mathrm{TT}}(\mathbf{k}, t) \approx 16\pi G \, \Lambda_{ij,lm} \int_0^t \mathrm{d}t' \, \frac{\sin[k(t - t')]}{k} T_{lm}(0, t')$$

Useful insight 1

- If the source $T_{lm}(0,t)$ varies faster than the $\sin[k(t-t')]$, the equation reduces to

$$h_{ij}^{\mathrm{TT}}(\mathbf{k},t) \approx 16\pi G\,\Lambda_{ij,lm}\int_0^t \mathrm{d}t'\,T_{lm}(0,t')$$

  This gives

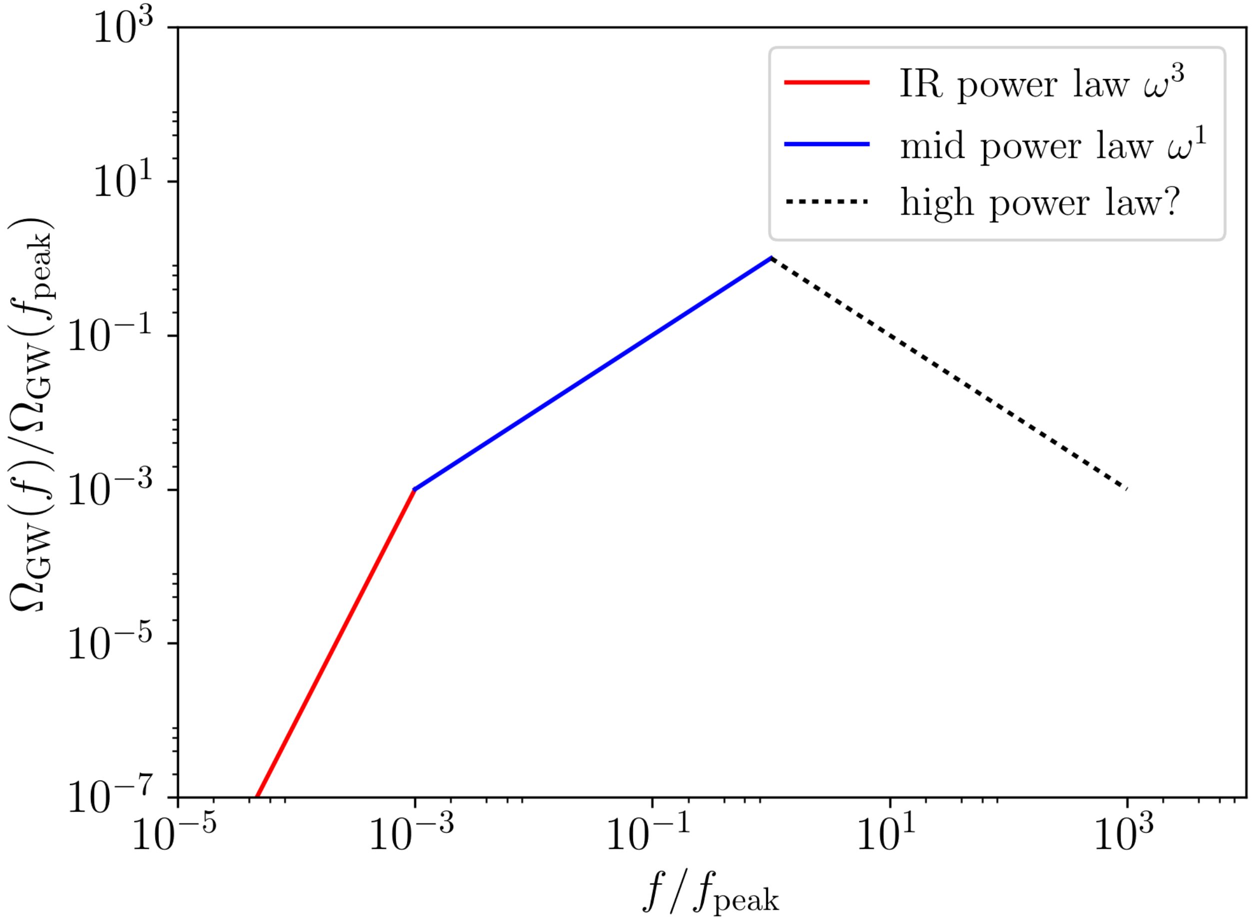
$$\frac{\mathrm{d}\rho_{\mathrm{GW}}(k)}{\mathrm{d}\ \log\ k} \propto k^3$$

- In other words, at **sufficiently long sub-horizon scales**, the quadrupole approximation always works and all the matters is **how long the source is on for**.
- The power law is k^3.
- This holds for e.g. first order phase transitions and the end of inflation.

Useful insight 2

- If the source has an *intermediate regime* where $T_{lm}(0,t)$ varies *slower* than $\sin[k(t-t')]$, then the source stays in the integral, and there is an additional $1/k$ factor
- Therefore, **in some cases** we can expect a k^1 power law at higher wavenumbers than the k^3 is valid
- This is less generally true than the previous k^3 regime, so it might not be observed at all.
- In general, though, where a power law is seen in simulation results, it is worth seeing if the underlying physics is amenable to simpliciation!
- We will encounter more power laws later in these lectures...

Useful insight: graphical summary



Conclusions

- With pulsar timing arrays, space- and earth-based detectors, we now (or very soon) will view the gravitational wave sky from nanohertz through to kilohertz
- The basic equations of gravitational radiation share a lot of features with electromagnetism (or other gauge theories)
- There are some useful regimes that one can explore with only very limited knowledge of the form of a source of gravitational waves.